Dynamical Horizons: Energy, Angular Momentum, Fluxes and Balance Laws

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Dynamical horizons are considered in full, non-linear general relativity. Expressions of fluxes of energy and angular momentum carried by gravitational waves across these horizons are obtained. Fluxes are local, the energy flux is positive and change in the horizon area is related to these fluxes. The flux formulae also give rise to balance laws analogous to the ones obtained by Bondi and Sachs at null infinity and provide generalizations of the first and second laws of black hole mechanics.

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Black holes which are themselves in equilibrium but in possibly time-dependent space-times can be modelled by *isolated horizons* [1]. Over the past three years, properties of isolated horizons were studied in detail. In particular, the framework enabled one to extend the laws of black hole mechanics [2, 3] and has been used to extract physics from initial data of widely separated black holes [4] and from numerical simulations of the final phases of black hole collisions [1, 5]. The purpose of this letter is to outline a generalization of these ideas to fully dynamical situations in which matter and gravitational radiation can fall into black holes.

Our definition of a dynamical horizon is geared to practical applications in astrophysical situations, particularly those considered in numerical relativity.

I. DEFINITION AND NOTATION

Definition: A smooth, three-dimensional, space-like sub-manifold H in a space-time is said to be a *dynamical horizon* if it is foliated by a preferred family of 2-spheres such that, on each leaf S, the expansion $\theta_{(\ell)}$ of a null normal ℓ^a vanishes and the expansion $\theta_{(n)}$ of the other null normal n^a is strictly negative.

Thus, a dynamical horizon H is a 3-manifold which is foliated by marginally trapped 2-spheres. As shown below, the area of these 2-spheres necessarily increases. An example is provided by continuous segments of world tubes of apparent horizons that naturally arise in numerical evolutions of black holes. In contrast to event horizons, dynamical horizons can be located quasi-locally; knowledge of the full space-time is not required. The condition that H be space-like is implied by a stronger but physically reasonable restriction that the derivative of $\theta_{(\ell)}$ along n^a be negative [6]. Finally, the requirement that the leaves be topologically S^2 can be replaced by the weaker condition that they be compact. One can show that the topology of S is necessarily S^2 if the flux of matter or gravitational energy across H is non-zero. If these fluxes were to vanish identically, H would become isolated and replaced by a null, non-expanding horizon [2].

Dynamical horizons are closely related to Hayward's trapping horizons [6]. However, while the definition of trapping horizons imposes a condition on the derivative of $\theta_{(\ell)}$ off H, our conditions refer only to geometric quantities which are intrinsically defined on H. But in cases of physical interest, the additional condition would be satisfied and dynamical horizons will be future, outer trapping horizons. Nonetheless, our analysis and results differ considerably from those of Hayward's. While his framework is based on a 2+2 decomposition, ours will be based on the ADM 3+1 decomposition. Our discussion includes angular momentum, our flux formulae are new and our generalization of black hole mechanics is different. Our analysis is geared to providing tools to extract physics and perform checks on numerical simulations of dynamical black holes. Therefore we will restrict ourselves to dynamical horizons with zero charge.

Let us begin by fixing notation. Let $\widehat{\tau}^a$ be the unit time-like normal to H and denote by ∇_a the space-time derivative operator. The metric and extrinsic curvature of H are denoted by q_{ab} and $K_{ab} := q_a{}^c q_b{}^d \nabla_c \widehat{\tau}_d$ respectively; D_a is the derivative operator on H compatible with q_{ab} and $\mathcal{R}_{ab}$ its Ricci tensor. Leaves of the preferred foliation of H will be called *cross-sections* of H. The unit space-like vector orthogonal to S and tangent to H is denoted by $\widehat{r}^a$. Quantities intrinsic to S will be generally written with a tilde. Thus, the two-metric on S is $\widetilde{q}_{ab}$, the extrinsic curvature of $S \subset H$ is $\widetilde{K}_{ab} := \widetilde{q}_a{}^c \widetilde{q}_b{}^d D_c \widehat{r}_d$, the derivative operator on $(S, \widetilde{q}_{ab})$ is $\widetilde{D}_a$ and its Ricci tensor is $\widetilde{\mathcal{R}}_{ab}$. Finally, we will fix the rescaling freedom in the choice of null normals via $\ell^a := \widehat{\tau}^a + \widehat{r}^a$ and $n^a := \widehat{\tau}^a - \widehat{r}^a$.

We first note an immediate consequence of the definition. Since $\theta_{(\ell)} = 0$ and $\theta_{(n)} < 0$, it follows that $\widetilde{K} > 0$. Hence the area a_S of S increases monotonically along $\widehat{r}^a$. Thus the second law of black hole mechanics holds on H. We will obtain an explicit expression for the change of area in part III.

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Our main analysis is based on the fact that, since H is a space-like surface, the Cauchy data (q_{ab}, K_{ab}) on H must satisfy the usual scalar and vector constraints

$$\begin{aligned} H_S &:= \mathcal{R} + K^2 - K^{ab}K_{ab} = 16\pi G T_{ab}\widehat{\tau}^a\widehat{\tau}^b & (1) \\ H_V^a &:= D_b\left(K^{ab} - Kq^{ab}\right) = 8\pi G T^{bc}\widehat{\tau}_c q^a{}_b\,. & (2) \end{aligned}$$

We will often focus our attention on a portion $\Delta H \subset H$ which is bounded by two cross-sections S_1 and S_2.

II. ANGULAR MOMENTUM

The angular momentum analysis is rather straight forward and is, in fact, applicable to an arbitrary space-like hypersurface. Fix *any* vector field φ^a on H which is tangential to the cross-sections of H. Contract φ^a with both sides of eqn. (2). Integrate the resulting equation over the region $\Delta H \subset H$, perform an integration by parts and use the identity $\mathcal{L}_\varphi q_{ab} = 2D_{(a}\varphi_{b)}$ to obtain

$$\frac{1}{8\pi G}\oint_{S_2} K_{ab}\varphi^a\widehat{r}^b\,d^2V - \frac{1}{8\pi G}\oint_{S_1} K_{ab}\varphi^a\widehat{r}^b\,d^2V = \int_{\Delta H}\left(T_{ab}\widehat{\tau}^a\varphi^b + \frac{1}{16\pi G}P^{ab}\mathcal{L}_\varphi q_{ab}\right)d^3V \qquad (3)$$

where $P^{ab} := K^{ab} - Kq^{ab}$. It is natural to identify the surface integrals with the generalized angular momentum $J^{(\varphi)}$ associated with those surfaces and set:

$$J_S^{(\varphi)} = -\frac{1}{8\pi G}\oint_S K_{ab}\varphi^a\widehat{r}^b\,d^2V \qquad (4)$$

where we have chosen the overall sign to ensure compatibility with conventions normally used in the asymptotically flat context. The term 'generalized' emphasizes the fact that the vector field φ^a need not be an axial Killing field even on S; it only has to be tangential to our cross-sections.

The flux of this angular momentum due to matter fields and gravitational waves are respectively

$$\begin{aligned} \mathcal{J}_{\rm m}^{(\varphi)} &= -\int_{\Delta H} T_{ab}\widehat{\tau}^a\varphi^b\,d^3V\,, & (5) \\ \mathcal{J}_{\rm g}^{(\varphi)} &= -\frac{1}{16\pi G}\int_{\Delta H} P^{ab}\mathcal{L}_\varphi q_{ab}\,d^3V\,, & (6) \end{aligned}$$

and we get the balance equation

$$J_2^{(\varphi)} - J_1^{(\varphi)} = \mathcal{J}_{\rm m}^{(\varphi)} + \mathcal{J}_{\rm g}^{(\varphi)}\,. \qquad (7)$$

As expected, if φ^a is a Killing vector of the three-metric q_{ab}, then the gravitational flux vanishes: $\mathcal{J}_{\rm g}^{(\varphi)} = 0$. It is convenient to introduce the *angular momentum current* $j^\varphi := -K_{ab}\varphi^a\widehat{r}^b$ so that (4) becomes $J_S^{(\varphi)} = (8\pi G)^{-1}\oint_S j^\varphi\,d^2V$.

III. ENERGY FLUXES AND AREA BALANCE

As is usual in general relativity, the notion of energy is tied to a choice of a vector field. Here, we will consider vector fields $\xi^a = N\ell^a$ where the lapse N is constructed as follows. Let r be a radial coordinate on H defined such that the cross-sections of H are level surfaces of r. Then $\widehat{r}_a \propto D_a r$. It turns out that in order to get the balance law for energy, we must tie our lapse functions N to radial coordinates such that $D_a r = N_r\widehat{r}_a$. (Since $\xi^a = N\ell^a$, as usual the term 'lapse' refers to space-time evolution; not to 'evolution' along $\hat{r}^a$.) Thus each r determines a *permissible* lapse function N_r. If we use a different radial coordinate r', then the lapse is rescaled according to the relation

$$N_{r'} = N_r\frac{dr'}{dr}\,. \qquad (8)$$

Thus, although the lapse itself will in general be a function of all three coordinates on H, the relative factor between any two permissible lapses can be a function only of r. We denote the resulting *permissible* vector fields by $\xi^a_{(r)} := N_r\ell^a$. Recall that, on an isolated horizon, physical fields are time independent and null normals can be rescaled by a positive constant [2]. Now the horizon fields are 'dynamical', i.e., r-dependent, and the rescaling freedom is by a positive function of r.

We are interested in calculating the flux of energy associated with $\xi^a_{(r)}$ for any radial coordinate r. Denote the flux of matter energy across ΔH by $\mathcal{F}_m^{(r)} := \int_{\Delta H} T_{ab}\widehat{\tau}^a\xi^b_{(r)}d^3V$. By taking the appropriate combination of (1) and (2) we obtain

$$\mathcal{F}_m^{(r)} = \frac{1}{16\pi G}\int_{\Delta H} N_r\left\{H_S + 2\widehat{r}_a H_V^a\right\}d^3V\,. \qquad (9)$$

Since H is foliated by two-spheres, we can perform a 2+1 split of the various quantities on H. Using the Gauss Codazzi relation we rewrite $\mathcal{R}$ in terms of quantities on S:

$$\mathcal{R} = \widetilde{\mathcal{R}} + \widetilde{K}^2 - \widetilde{K}_{ab}\widetilde{K}^{ab} + 2D_a\alpha^a \qquad (10)$$

where $\alpha^a = \widehat{r}^bD_b\widehat{r}^a - \widehat{r}^aD_b\widehat{r}^b$. Next, the fact that the expansion $\theta_{(\ell)}$ of ℓ^a vanishes leads to the relation

$$K + \widetilde{K} = K_{ab}\widehat{r}^a\widehat{r}^b\,. \qquad (11)$$

Using (10) and (11) in eqn. (9) and simplifying, we obtain the result

$$\int_{\Delta H} N_r\widetilde{\mathcal{R}}\,d^3V = 16\pi G\int_{\Delta H} T_{ab}\widehat{\tau}^a\xi^b_{(r)}\,d^3V + \int_{\Delta H} N_r\left\{|\sigma|^2 + 2|\zeta|^2\right\}d^3V \qquad (12)$$

where $|\sigma|^2 = \sigma_{ab}\sigma^{ab}$ with $\sigma_{ab} := \widetilde{q}_a{}^c\widetilde{q}_b{}^d\nabla_c\ell_d - \frac{1}{2}\widetilde{q}_{ab}\widetilde{q}^{\,cd}\nabla_c\ell_d$, the shear of ℓ^a, and $|\zeta|^2 = \zeta^a\zeta_a$ with

$\zeta^a := \widetilde{q}^{ab}\widehat{r}^c\nabla_c\ell_b$. Both σ_{ab} and ζ^a are tensors intrinsic to S. To simplify the left side of this equation, note that the volume element d^3V on H can be written as $d^3V = N_r^{-1}dr\, d^2V$ where d^2V is the area element on S. Using the Gauss-Bonnet theorem, the integral of $N_r\widetilde{\mathcal{R}}$ can then be written as

$$\int_{\Delta H} N_r\widetilde{\mathcal{R}}\, d^3V = \int_{r_1}^{r_2} dr\left(\oint_S \widetilde{\mathcal{R}}\, d^2V\right) = 8\pi(r_2 - r_1)\,. \tag{13}$$

Substituting this result in eqn. (12) we finally obtain

$$\begin{aligned}\left(\frac{r_2}{2G} - \frac{r_1}{2G}\right) &= \int_{\Delta H} T_{ab}\widehat{\tau}^a\xi^b_{(r)}\, d^3V \\ &+ \frac{1}{16\pi G}\int_{\Delta H} N_r\left\{|\sigma|^2 + 2|\zeta|^2\right\}\, d^3V.\end{aligned} \tag{14}$$

This is the key result we were looking for. Let us now interpret the various terms appearing in this equation. The first integral on the right side of this equation is the flux $\mathcal{F}^{(r)}_m$ of matter energy associated with the vector field $\xi^a_{(r)}$. Since $\xi^a_{(r)}$ is null and $\widehat{\tau}$ time-like, if T_{ab} satisfies, say, the dominant energy condition, this quantity is guaranteed to be non-negative. It is natural to interpret the second term as the flux $\mathcal{F}^{(r)}_g$ of $\xi^a_{(r)}$-energy in the gravitational radiation:

$$\mathcal{F}^{(r)}_g := \frac{1}{16\pi G}\int_{\Delta H} N_r\left\{|\sigma|^2 + 2|\zeta|^2\right\}\, d^3V\,. \tag{15}$$

This expression shares four desirable features with the Bondi-Sachs energy flux at null infinity. First, it does not refer to any coordinates or tetrads; it refers only to the given dynamical horizon H and the evolution vector field $\xi^a_{(r)}$. Second, the energy flux is manifestly non-negative. Third, all fields used in it are local; we did not have to perform, e.g., a radial integration to define any of them. Finally, the expression vanishes in the spherically symmetric case: if the Cauchy data (q_{ab}, K_{ab}) and the foliation on H is spherically symmetric, $\sigma_{ab} = 0$ and $\zeta^a = 0$.

To conclude this section, let us choose for our radial coordinate the area radius $R := \sqrt{a/4\pi}$. Then,

$$\frac{R_2}{2G} - \frac{R_1}{2G} = \mathcal{F}^{(R)}_m + \mathcal{F}^{(R)}_g \tag{16}$$

Thus, as promised in part I, we have obtained an explicit formula relating the change in the area of the horizon to fluxes of matter and gravitational $\xi_{(R)}$-energy.

IV. MASS AND THE FIRST LAW

Let us now combine the results of parts II and III to obtain the physical process version of the first law for H and a mass formula for an arbitrary cross-section of H.

Denote by $E^{\xi_{(R)}}$ the $\xi_{(R)}$-energy of cross-sections S of H. While we do not yet have the explicit expression for it, we can assume that, because of the influx of matter and gravitational energy, $E^{\xi_{(R)}}$ will change by an amount $\Delta\, E^{\xi_{(R)}} = \mathcal{F}^{(R)}_m + \mathcal{F}^{(R)}_g$ as we move from one cross section to another. Therefore, if we define *effective surface gravity* k_R associated with $\xi^a_{(R)}$ as $k_R := 1/2R$, the infinitesimal form of (16) implies $(k_R/8\pi G)da = dE^{\xi_{(R)}}$. For a general choice of the radial coordinate r, (14) yields a generalized first law:

$$\frac{k_r}{8\pi G}da = dE^{\xi_{(r)}} \tag{17}$$

where the effective surface gravity k_r of $\xi^a_{(R)}$ is given by

$$k_r = \frac{dr}{dR}k_R \qquad \text{where} \qquad \xi^a_{(r)} = \frac{dr}{dR}\xi^a_{(R)}\,. \tag{18}$$

This rescaling freedom in surface gravity is analogous to the rescaling freedom which exists for Killing horizons, or more generally, isolated horizons. The new feature in the present case is that we have the freedom to rescale the surface gravity (and ℓ^a) by a positive function of the radius instead of just by a constant. This is just what one would expect in a dynamical situation. Finally, note that the differentials appearing in (17) are actual variations along the dynamical horizon due to an infinitesimal change in r and are not variations in phase space as in some of the formulations [2, 3, 7] of the first law.

To include rotation, pick a vector field φ^a on H such that φ^a is tangent to the cross-sections of H, has closed orbits and has affine length 2π.(At this point, φ^a need not be a Killing vector of q_{ab}.) Consider time evolution vector fields t^a which are of the form $t^a = N_r\ell^a - \Omega\varphi^a$ where N_r is a permissible lapse associated with a radial coordinate r and Ω an arbitrary function of r. Evaluate the quantity $\int_{\Delta H} T_{ab}\widehat{\tau}^a t^b\, d^3V$ using (3) and (14):

$$\begin{aligned}&\frac{r_2 - r_1}{2G} + \frac{1}{8\pi G}\left\{\oint_{S_2}\Omega j^\varphi\, d^2V - \oint_{S_1}\Omega j^\varphi\, d^2V\right. \\ &\left.-\int_{\Omega_1}^{\Omega_2} d\Omega\oint_S j^\varphi\, d^2V\right\} = \int_{\Delta H} T_{ab}\widehat{\tau}^a t^b\, d^3V \\ &+\frac{1}{16\pi G}\int_{\Delta H} N_r\left(|\sigma|^2 + 2|\zeta|^2\right)\, d^3V \\ &-\frac{1}{16\pi G}\int_{\Delta H}\Omega P^{ab}\mathcal{L}_\varphi q_{ab}\, d^3V\,.\end{aligned} \tag{19}$$

Again, if we denote by E^t the t-energy associated with cross-sections S of H, the right side of (19) can be interpreted as ΔE^t. If we now restrict ourselves to infinitesimal ΔH, the three terms in the curly brackets combine to give $d(\Omega J) - Jd\Omega$ and we obtain

$$\frac{dr}{2G} + \Omega dJ = \frac{k_r}{8\pi G}da + \Omega dJ \,=\, dE^t. \tag{20}$$

This equation is our generalization of the first law for dynamical horizons. Since the differentials in this equation are variations along H, this can be viewed as a 'physical process version of the first law'. Note that for each

allowed choice of lapse N_r, angular velocity $\Omega(r)$ and vector field φ^a on H, we obtain a permissible time evolution vector field $t^a = N_r\ell^a - \Omega\varphi^a$ and a corresponding first law. This situation is very similar to what happens in the isolated horizon framework where we obtain a first law for each permissible time translation on the horizon. Again, the generalization from that time independent situation consists of allowing the lapse and the angular velocity to become r-dependent, i.e., 'dynamical'.

For every allowed choice of $(N_r, \Omega(r), \varphi^a)$, we can integrate eqn. (20) on H to obtain a formula for E^t on any cross section but, in general, the result may not be expressible just in terms of geometric quantities defined locally on that cross-section. However, in some physically interesting cases, the expression *is* local. For example, In the case of spherical symmetry, it is natural to choose $\Omega = 0$ and R as the radial coordinate in which case we obtain $E^t = R/2G$. This is just the irreducible (or Hawking) mass of the cross-section. Even in this simple case, (19) provides a useful balance law, with clear-cut interpretation. Physically, perhaps the most interesting case is the one in which q_{ab} is only axi-symmetric with φ^a as its axial Killing vector. In this case we can naturally apply, at each cross-section S of H, the strategy used in the isolated horizon framework to select a preferred t^a: Calculate the angular momentum J defined by the axial Killing field φ, choose the radial coordinate r (or equivalently, the lapse N_r) such that

$$k_r = k_o(R) := \frac{R^4 - 4G^2J^2}{2R^3\sqrt{R^4 + 4G^2J^2}} \tag{21}$$

and choose Ω such that

$$\Omega = \Omega_o(R) := \frac{2GJ}{R\sqrt{R^4 + 4G^2J^2}}\,. \tag{22}$$

This functional dependence of k_r on R and J is exactly that of the Kerr family. (The condition on surface gravity can always be implemented provided the right side of (21) is positive, which in the kerr family corresponds to non-extremal horizons. The resulting r and N_r are unique.) With this choice of N_r and Ω, the energy E^t_S is given by the well known Smarr formula

$$E^{t_o} = 2\,(\frac{k_o a}{8\pi G} + \Omega_o J) = \frac{\sqrt{R^4 + 4G^2J^2}}{2GR}\,; \tag{23}$$

as a function of its angular momentum and area, each cross-section is assigned simply that mass which it would have in the Kerr family. However, there is still a balance equation in which the flux of gravitational energy $\mathcal{F}_g^{(t_o)}$ is local and positive definite (see (19)). (The gravitational angular momentum flux which, in general, has indeterminate sign vanishes due to axi-symmetry.) Motivated by the isolated horizon framework, we will refer to this canonical E^{t_o} as the *mass* associated with cross-sections S of H and denote it simply by M. Thus, among the infinitely many first laws (20), there is a canonical one:

$$dM = \frac{k_o}{8\pi G}da + \Omega_o dJ\,. \tag{24}$$

We conclude with three remarks.

i) Note that the mass and angular momentum depend only on *local* fields on each cross section S and changes in these quantities over *finite* regions ΔH of H have been related to matter and gravitational radiation fluxes, determined by the *local* geometry of H.

ii) Unlike the vector fields $\xi^a_{(r)} = N_r\ell^a$, general permissible vector fields t^a is not necessarily causal. Therefore the matter flux $\int_{\Delta H} T_{ab}t^a\widehat{\tau}^b d^3V$ need not be positive. Similarly, if φ^a is not a Killing field of q_{ab}, the gravitational flux need not be positive. Therefore, although the area a always increase along $\hat{r}^a$, E^t can decrease. This is the analog of the Penrose process in which 'rotational energy' is extracted from the dynamical horizon.

iii) While the infinitesimal version eq (20) of the first law is conceptually more interesting, the finite balance equation (19) is likely to be more directly useful in the analysis of astrophysical situations. In particular, the presence of an infinite number of these balance equations can provide useful checks on numerical simulations in the strong field regime.

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